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The paper is concerned with counterparty credit risk management for credit default swaps in the presence of default contagion. In particular, we study the impact of default contagion on credit value adjustments such as the BCCVA (Bilateral Collateralized Credit Value Adjustment) of (Brigo et al. 2012) and on the performance of various collateralization strategies. We use the incomplete-information model of Frey and Schmidt (2012) as vehicle for our analysis. We find that taking contagion effects into account is important for the effectiveness of the strategy and we derive refined collateralization strategies to account for contagion effects.

Keywords: Counterparty credit risk; bilateral credit value adjustment; contagion; credit default swap; incomplete information; collateralization.

1. Introduction

The distress of many financial firms in recent years made counterparty risk management for OTC derivatives such as credit default swaps (abbreviated CDS) an issue of high concern in financial risk management. Crucial tasks in that context are the computation of *credit value adjustments* that account for the possibility that one of the contracting parties defaults before the maturity of the OTC contract, and the mitigation of counterparty risk by *collateralization*. Collateralization refers to the practice of posting securities (the so-called collateral) that serve as a pledge for the collateral taker. These securities are liquidated if one of the contracting parties defaults before maturity, and the proceeds are used to cover the replacement cost of the contract. In order to ensure that the funds generated in this way are sufficient, the collateral position needs to be adjusted dynamically in reaction to changes in the value of the underlying derivative security. The price dynamics of that security thus play a crucial role for the performance of a given collateralization strategy.

In the present paper we study the impact of different price dynamics on the size of value adjustments and on the performance of collateralization strategies for credit default swaps. We are particularly interested in the influence of contagion. Contagion effects - the fact that the default of a firm leads to a sudden increase in the credit spread of surviving firms - are frequently observed on real markets; a prime example are the events that surrounded the default of Lehman Brothers in 2008. To see that contagion might be relevant for the performance of collateralization strategies consider the scenario where the protection seller defaults during the runtime of the CDS. In that case contagion might lead to a substantial increase in the credit spread of the reference entity (the firm on which the CDS is written) and hence in turn to a much higher replacement value for the CDS. In standard collateralization strategies this is taken into account at most in a very crude way, and the amount of collateral posted before the default might be insufficient for replacing the CDS. In our view this issue merits a detailed analysis in the context of dynamic portfolio credit risk models.

We use the reduced-form credit risk model proposed by Frey and Schmidt (2012) as vehicle for our analysis. In that model it is assumed that the default times of the reference entity, the protection seller and the protection buyer are conditionally independent given some finite state Markov chain X that models the economic environment. We consider two versions of the model which differ with respect to the information that is available for investors. In the full-information model it is assumed that X is observable and there are no contagion effects. In the incomplete-information version of the model investors observe the chain X in additive Gaussian noise and moreover the default history. In that case there is default contagion that is caused by the updating of the conditional distribution of X_t at default events. An advantage of the setup of Frey and Schmidt (2012) for our purposes is the fact that the the joint distribution of the default times is the same in the two versions of the model. Hence differences in the size of value adjustments or in the performance of collateralization strategies can be attributed to the different dynamics of credit spreads (contagion or no contagion) in the two model variants.

In order to compute value adjustments and to measure the performance of collateralization strategies we use the bilateral collateralized credit value adjustment (BCCVA) proposed by Brigo et al. (2012). This credit value adjustment accounts for the form of collateralization strategies and for the credit quality of the contracting parties. Our analysis reveals that the impact of contagion on the size of the BCCVA depends strongly on the relative credit quality of the three parties involved and is hard to predict up front. Results on the performance of different collateralization strategies are more clear-cut: we show that while standard market-value based collateralization strategies provide a good protection against losses due to counterparty risk in the full-information setup, they have problems to deal with the contagious jump in credit spreads at a default of the protection seller. Motivated by these findings we go on and develop an improved collateralization strategy that performs well in the presence of contagion. For our analysis we need to compute the BCCVA in both model variants. Here use the theory of phase-type distributions to derive new explicit formulas for the BCCVA under full information; this is the main mathematical contribution of the paper.

There is by now a large literature on counterparty risk for CDSs. Existing contributions focus mostly on the computation of value adjustments (with and without collateralization) in various credit risk models. Counterparty credit risk and valuation adjustments for uncollateralized CDS are studied by Hull and White (2001), Brigo and Chourdakis (2009), Blanchet-Scalliet and Patras (2008), Lipton and Sepp (2009) and Bao et al. (2010), among others. Counterparty credit risk for collateralized CDS is discussed in Bielecki et al. (2011), Fujii and Takahashi (2011) and Brigo et al. (2012).

However, none of these contributions covers the issues discussed in this paper in full. Bielecki et al. (2011) analyze the impact of collateralization on counterparty risk in CDS contracts using the Markov copula model which does not exhibit contagion effects whereas Fujii and Takahashi (2011) is essentially concerned with the impact of collateralization on the price of a CDS that is subject to counterparty risk. Brigo et al. (2012) is closest to our contribution: these authors study the impact of contagion on credit value adjustments and on the effectiveness of market-value based collateralization strategies in a Gaussian copula model with stochastic credit spreads. In that model default or event correlation and contagion effects are both driven by the choice of the correlation parameter of the copula. Consequently, it is not possible to disentangle the impact of event correlation and of default contagion on credit value adjustments and on the performance of collateralization strategies. This might be an advantage of our setup. Moreover, Brigo et al. (2012) do not address the issue of designing collateralization strategies that take default contagion into account.

The remainder of the paper is organized in the following way. In Section 2 we discuss the BCCVA of Brigo et al. (2012). In Section 3 we introduce the credit risk model of Frey and Schmidt (2012) that provides the framework for the analysis of the present paper. Section 4 is devoted to the computation of the BCCVA in both model variants. In Section 5 we discuss different collateralization strategies, and in Section 6 we present the results of a simulation study.

2. Bilateral Collateralized Credit Value Adjustment

In this section we discuss the bilateral collateralized credit value adjustment (BCCVA) proposed in Brigo et al. (2012). We begin with some notation. Throughout the entire paper we work on a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ equipped with a filtration $\mathbb{F} := (\mathcal{F}_t)_{t\in[0,T]}$ that fulfills the usual hypotheses. $\mathbb{Q}$ denotes the risk-neutral measure used for pricing, and all expectations are taken with respect to $\mathbb{Q}$. The filtration $\mathbb{F}$ contains the information, which is available to the market participants and will be specified later when the model framework is discussed. Moreover, we assume that interest rates are constant and equal to r and we denote the discount factor from time t to time s by $D(t, s) = e^{-r(t-s)}$.

The following parties are involved in the CDS contract: the protection buyer, labeled B; the reference identity, labeled R; the protection seller, labeled S. The default times of these entities are denoted by τ_B, τ_R and τ_S. We assume that defaults are observable so that τ_B, τ_R and τ_S are $\mathbb{F}$ stopping times. The first default time is denoted by τ, that is $\tau := \tau_B \wedge \tau_R \wedge \tau_S$. The random variable ξ with values in the set $\{B, R, S\}$ represents the identity of the firm defaulting at τ. Furthermore let Rec_B, Rec_R, Rec_S denote the recovery rate and LGD_B, LGD_R, LGD_S the loss given default of of B, R and S. We assume that recovery rates are constant.

All valuations and cash flows are defined from the perspective of the protection buyer. Therefore positive numbers are used to indicate that a cash flow is received by the protection buyer and negative numbers to indicate that a cash flow is received by the protection seller.

Payments of a risk-free CDS. In our context a CDS without counterparty risk, which we call (counterparty-) risk-free CDS, is a CDS where neither the protection seller nor the protection buyer are subject to default risk. For simplicity, we assume that the premium payments are paid continuously. Therefore the sum of all discounted payments in a risk-free CDS from time t to time s discounted to t, denoted by $\Pi(t, s)$ is given by:

$$\Pi(t,s) := 1_{\{t<\tau_R\le s\}}\,\mathrm{LGD}_R\,D(t,\tau_R) - \int_t^s S_R D(t,u) 1_{\{\tau_R>u\}} du. \tag{2.1}$$

Here S_R represents the spread of the CDS. In addition we define the price at time t P_t of a risk-free CDS with maturity date $T > t$ as the risk-neutral expectation of $\Pi(t, T)$, that is

$$P_t := \mathbb{E}\left(\Pi(t,T)|\mathcal{F}_t\right).$$

Risky CDS and collateralization. In a CDS with counterparty risk, called *risky CDS* below, the protection buyer or the protection seller might default before the maturity of the CDS. Collateralization is a way to limit the potential loss for the surviving party. The details of the collateralization procedure are stipulated in the credit support annex (CSA) of the contract. A simplified version of the procedure works as follows. At $t_0 = 0$ a collateral account is opened. Let C_t denote the cash balance[a] in the account at time t, where $C_t > 0$ ($C_t < 0$) means that the protection seller (the protection buyer) has posted the collateral and that the protection buyer (the protection seller) is the collateral taker. The collateral taker has unrestricted access to the posted collateral. The collateral is updated at discrete time points t_0, $t_1, \dots, t_N = T$ that are specified in the CSA. At t_1 the collateral taker pays interest on the collateral and the collateral amount is updated in reaction to changes in the price of the underlying CDS over $(t_0, t_1]$ as stipulated in the CSA. This procedure

[a]In this paper we assume that the collateral is posted in form of cash and not in form of other assets.

continues up to the maturity of the CDS or until the first default occurs. If $\tau > T$ or if $\tau < T$ and $\xi = R$ the collateral account is closed at the 'natural end' of the contract so that $C_t \equiv 0$ for $t \geq \tau \wedge T$ in that case. If there is an early default of B or S, that is for $\tau \leq T$ and $\xi \in \{B, S\}$, the collateral is used to reduce the exposure of the collateral taker and any remaining collateral is returned; details are given below.

An issue arising in this context is *re-hypothecation*: since the collateral taker has unrestricted access to the posted collateral it may happen that a part of the collateral is lost as a consequence of the default event. We denote by Rec'_B and Rec'_S the recovery rate for the return of collateral and by LGD'_B and LGD'_S the corresponding loss given default (assumed constant). Usually the return of collateral is favored to other claims in bankruptcy procedures, so that $\mathrm{Rec}_B \leq \mathrm{Rec}'_B$ and $\mathrm{Rec}_S \leq \mathrm{Rec}'_S$. Contracts without re-hypothecation are characterized by $\mathrm{Rec}'_B = 1$ and $\mathrm{Rec}'_S = 1$.

Formally, we model the cash balance in the collateral account at time t by some adapted RCLL process $C = (C_t)_{0 \leq t \leq T}$ (the collateralization strategy). For simplicity we assume that interest on the collateral amount is paid continuously. Hence from the perspective of the protection buyer collateralization leads to a cumulative cash flow stream of the form $C_t - \int_0^t rC_s ds$, $t \leq T$. By partial integration the discounted value of that cash-flow stream at $t = 0$ equals

$$\Pi^C(0,T) = C_0 + \int_0^T e^{-rs} dD_s = \int_0^T e^{-rs} dC_s + \int_0^T -re^{-rs} C_s ds = e^{-rT} C_T \, . \quad (2.2)$$

Recall that $C_T = 0$ on $\{\tau > T\}$ and on $\{\tau < T\} \cap \{\xi = R\}$. Hence scenarios where the CDS ends naturally can be ignored in the computation of the counterparty risk, and it suffices to consider the collateral payments for the case where there is an early default of R or S. Note that in this argument we have used the risk-free rate r for discounting the cash-flow stream D, thus ignoring the funding cost of the collateral. We think that this simplification can be justified as funding costs are not central to the objectives of this paper. For an analysis of credit value adjustments in the presence of funding cost we refer to Brigo et al. (2011b), Crepey (2012a) and Crepey (2012b).

Payments at an early default. In order to complete the description of the cash flow stream of a risky CDS we need to specify the payments at an early default of B or S, that is for $\tau \leq T$ and $\xi \in \{B, S\}$. In that case the surviving party is allowed to charge a *close-out amount* from the defaulting one. According to the ISDA Master Agreement the close-out amount is defined as the reasonable amount of money which is needed to close the position. In this paper we assume that the close-out amount is given by P_τ, the value of the risk-free CDS at the first default time. Note that this choice means that the credit quality of the surviving party is completely neglected in the computation of the close-out amount, which is in line with current market practice. However, there are alternative suggestions in the literature; see for instance Brigo et al. (2012a).

We continue with the description of the payments at an early default. To shorten the exposition we concentrate on the payments in the case where the protection seller defaults first. Note that no additional collateral is posted after the default of S. Hence we assume that the amount of collateral available during the bankruptcy process is given by $C_{\tau-}$ (the amount of collateral that has been posted immediately prior to τ). This distinction matters if the close-out amount P_t jumps at $t = \tau$, for instance due to contagion effects.

We have to consider four scenarios that differ with respect to the sign of P_τ and of $C_{\tau-}$.

(1) Suppose that $P_\tau > 0$ and that the protection buyer is the collateral taker, that is $C_{\tau-} > 0$. The collateral is used to reduce the loss of the protection buyer. If $C_{\tau-}$ exceeds P_τ, the excess collateral is returned to the protection seller. If $C_{\tau-}$ is smaller than P_τ, the protection buyer claims the difference $P_\tau - C_{\tau-}$ from S. However, B will receive only a recovery payment of size $\mathrm{Rec}_S(P_\tau - C_{\tau-})$ in that case. With the notation $X^+ := \max(X, 0)$ and $X^- := -\min(X, 0)$,[b] the overall payment at τ takes the form:

$$1_{\{\tau<T\}}1_{\{\xi=S\}}1_{\{P_\tau>0\}}1_{\{C_{\tau-}>0\}}\big(\mathrm{Rec}_S(P_\tau - C_{\tau-})^+ - (P_\tau - C_{\tau-})^-\big).$$

(2) Consider the case when $P_\tau > 0$ and $C_{\tau-} < 0$, so that the protection seller is the collateral taker. In this situation B is entitled to the repayment of the collateral and to the close-out amount P_τ. However, only a fraction of P_τ and, due to re-hypothecation, of $C_{\tau-}$ will be paid to B. Hence the overall payment takes the form

$$1_{\{\tau<T\}}1_{\{\xi=S\}}1_{\{P_\tau>0\}}1_{\{C_{\tau-}<0\}}\big(\mathrm{Rec}_S\, P_\tau - \mathrm{Rec}'_S\, C_{\tau-}\big).$$

(3) Suppose that $P_\tau < 0$ and that the protection buyer is the collateral taker, that is $C_{\tau-} > 0$. In that case B pays S the close-out amount P_τ and he returns the collateral. Hence from the viewpoint of B, the overall payment equals

$$1_{\{\tau<T\}}1_{\{\xi=S\}}1_{\{P_\tau<0\}}1_{\{C_{\tau-}>0\}}\big(P_\tau - C_{\tau-}\big).$$

(4) Suppose that $P_\tau < 0$ and that B posted some collateral so that $C_{\tau-} < 0$. If $-C_{\tau-} \le -P_\tau$ S keeps the collateral and he moreover receives the difference $-(P_\tau - C_\tau)$. Otherwise the excess collateral has to be returned to B, and there might be losses due to re-hypothecation. Hence the overall payment in that case equals

$$1_{\{\tau<T\}}1_{\{\xi=S\}}1_{\{P_\tau<0\}}1_{\{C_{\tau-}<0\}}\big(\mathrm{Rec}'_S(P_\tau - C_{\tau-})^+ - (P_\tau - C_{\tau-})^-\big).$$

The payments that arise if the protection buyer defaults first, that is for $\xi = B$, can be described in an analogous manner.

[b]Note that the convention $X^- := \min(X, 0)$ is used in Brigo et al. (2011a) and Brigo et al. (2012).

The BCCVA. Given a collateralization strategy C, the *bilateral collateralized credit value adjustment* (BCCVA) is defined as difference of the discounted cash-flow stream of the risk-free and the risky CDS. Following Brigo et al. (2012), we denote the latter cash-flow stream by $\Pi^D(t,T,C)$. We thus have

$$\mathrm{BCCVA}(t,T,C) := \mathbb{E}\left(\Pi(t,T)|\mathcal{F}_t\right) - \mathbb{E}\left(\Pi^D(t,T,C)\big|\mathcal{F}_t\right). \tag{2.3}$$

Using the above description of the payments at an early default it is straightforward to give an explicit formula for $\Pi^D(t,T,C)$. However, in this paper we use an expression for the BCCVA that does not involve Π^D directly (see Proposition 2.1 below) so that we omit the formula and refer to Brigo et al. (2012) instead.

By definition the BCCVA thus measures the difference in value of the cash-flows of a risk-free CDS and a risky CDS. Note that the BCCVA takes the risk of a default of S *and* of B into account. The BCCVA leads to symmetrical prices in the sense that the adjustment computed from the point of view of the protection buyer equals (with the opposite sign) the adjustment computed from the point of view of the protection seller.

In the sequel we work with the following representation of the BCCVA that is established in Brigo et al. (2012).

Proposition 2.1. *The* BCCVA *can be decomposed as follows*

$$\mathrm{BCCVA}(t,T,C) = \mathrm{CCVA}(t,T,C) - \mathrm{CDVA}(t,T,C), \tag{2.4}$$

where the collateralized credit value adjustment (CCVA) *and the* collateralized debt value adjustment (CDVA) *are given by:*

$$\begin{aligned}\mathrm{CCVA}(t,T,C) &:= \mathbb{E}\big(1_{\{\tau<T\}}1_{\{\xi=S\}}D(t,\tau)\ (\mathrm{LGD}_S(P_\tau^+ - C_{\tau-}^+)^+ \\ &\qquad\qquad + \mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)^+)|\mathcal{F}_t\big) \\ \mathrm{CDVA}(t,T,C) &:= \mathbb{E}\big(1_{\{\tau<T\}}1_{\{\xi=B\}}D(t,\tau)\ (\mathrm{LGD}_B(C_{\tau-}^- - P_\tau^-)^- \\ &\qquad\qquad + \mathrm{LGD}'_B(P_\tau^+ - C_{\tau-}^+)^-)|\mathcal{F}_t\big).\end{aligned}$$

Comments. 1. The CCVA reflects the possible loss for B due to an early default of S, whereas the CDVA reflects the loss of S due to an early default of B. Consider for instance the case where S defaults first. If $P_\tau > 0$, there are two reasons why B might incur a loss: first, the collateral posted by S might be insufficient to cover the close-out amount of the CDS, which leads to a loss of size $\mathrm{LGD}_S(P_\tau - C_{\tau-}^+)^+$; if $C_{\tau-} < 0$ there is moreover a loss due to re-hypothecation of size $\mathrm{LGD}'_S\, C_{\tau-}^-$. The overall loss thus corresponds to the argument of the CCVA-formula above as $P_\tau^- = 0$ for $P_\tau > 0$. If $P_\tau < 0$, B incurs a loss of size $\mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)$ (the loss of the excess collateral caused by re-hypothecation). Again this corresponds to the argument of the CCVA-formula in that case.

2. Without collateralization, that is for $C_t \equiv 0$, the CCVA and the CDVA take the form of options on the risk-free CDS price P with strike price $K = 0$ and random maturity date τ.

3. The Model

Next we give a mathematical description of the model framework we use in the remainder of this paper. We consider a reduced-form model where τ_R, τ_B and τ_S are conditionally independent, doubly-stochastic random times whose default intensity is driven by a finite-state Markov chain $X = (X_t)_{t\geq 0}$ with state space $S^X = \{1, 2, \dots, K\}$, generator matrix $W = (w_{ij})_{1\leq i,j\leq K}$ and initial distribution described by the probability vector π_0 with $\pi_0^k = \mathbb{Q}(X_0 = k)$. Denote by $\mathcal{F}_t^X := \sigma(X_s : s \leq t)$ the filtration generated by X. We assume that for all time points $t_1, t_2, t_3 > 0$ one has

$$\mathbb{Q}(\tau_R > t_1, \tau_B > t_2, \tau_S > t_3 \mid \mathcal{F}_\infty^X) = \prod_{i\in\{B,R,S\}} \exp\Big(- \int_0^{t_i} \lambda_i(X_s) ds\Big) \tag{3.1}$$

where $\lambda_i : S^X \to \mathbb{R}^+$, $i \in \{B, R, S\}$, are deterministic functions. We introduce the survival indicator functions $H_t^B := 1_{\{\tau_B > t\}}$, $H_t^R := 1_{\{\tau_R > t\}}$ and $H_t^S := 1_{\{\tau_S > t\}}$ and we put $H := (H^B, H^R, H^S)$ and $\mathcal{F}_t^H = \sigma(H_s : s \leq t)$. Moreover, we assume that the underlying probability space $(\Omega, \mathcal{F}, \mathbb{Q})$ supports a d-dimensional standard Brownian motion W which is independent of X and H; W is used to model investor information under imperfect observation of X (see below). The interpretation of this framework is easiest if $\lambda_B(\cdot)$, $\lambda_R(\cdot)$ and $\lambda_S(\cdot)$ are increasing in x. In that case the Markov chain X can be viewed as abstract representation of the state of the economy, 1 being the best state (low default probability of all firms) and K the worst state (high default probability of all firms).

In the sequel we will consider two variants of the model that differ with respect to the assumptions made on investor information.

The full-information case. Here it is assumed that X is *observable* for investors and we take $\mathbb{F} = \mathbb{F}^O$ with

$$\mathbb{F}^O = \mathbb{F}^H \vee \mathbb{F}^X \vee \mathbb{F}^W.$$

(The inclusion of $\mathbb{F}^W$ is purely technical and has no impact on the prices of credit derivatives under full information). It is well-known that for time points $t_R, t_S, t_B > t$ the conditional survival function given the investor information at time t satisfies

$$\mathbb{Q}(\tau_R > t_R, \tau_B > t_B, \tau_S > t_S \mid \mathcal{F}_t^O) = \prod_{i\in\{B,R,S\}} H_t^i \, \mathbb{E}\Big(\exp\Big(- \int_t^{t_i} \lambda_i(X_s) ds\Big) \mid X_t\Big) . \tag{3.2}$$

Moreover, the process $\lambda_i(X_t)$, $i \in \{B, R.S\}$ is the $\mathbb{F}^O$ default intensity of τ_i, and the pair process (X, H) is Markov. A derivation of these results can be found in Chapter 9 of McNeil, Frey and Embrechts (2005), among others. Formula (3.2) implies in particular that prior to the default of R the price of the risk-free CDS is a function of t and X_t,

$$P_t^O = \mathbb{E}\left(\Pi(t,T) \middle| \mathcal{F}_t^O\right) = p^O(t, X_t).$$

An explicit formula for the function p^O is given Section 4.1 below.

The incomplete-information case. This variant of the model has been studied in detail in Frey and Schmidt (2012). In that paper it is assumed that X is *unobservable* and that investors are confined to a noisy signal of X of the form

$$Z_t := \int_0^t a(X_s)ds + W_t\,,$$

where $a : S^X \to \mathbb{R}^d$ is a deterministic function. Hence we put $\mathbb{F} = \mathbb{F}^U$ with $\mathbb{F}^U = \mathbb{F}^H \vee \mathbb{F}^Z$. Note that $\mathbb{F}^U \subset \mathbb{F}^O$ by construction. Define the conditional probabilities

$$\pi_t^k := \mathbb{Q}(X_t = k \mid \mathcal{F}_t^U),\ 1 \le k \le K, \quad \text{and let } \pi_t := (\pi_t^1, \ldots, \pi_t^K)^\top. \tag{3.3}$$

As shown in Frey and Schmidt (2012), the process $(\pi_t)_{0\le t\le T}$ is the natural state variable process for the model under incomplete information. The reasons are the following: first, it is well-known that the intensity of H^i with respect to the subfiltration $\mathbb{F}^U$ is given by the projection of the intensity on $\mathbb{F}^U$ (see Chapter 2 of Bremaud (1981)), so that the $\mathbb{F}^U$-default intensities are given by

$$\widehat{\lambda_i(X_t)} := \mathbb{E}\left(\lambda_i(X_t)\big|\mathcal{F}_t^U\right) = \sum_{k=1}^K \lambda_i(k)\pi_t^k \quad i \in \{R, B, S\}. \tag{3.4}$$

Second, for $\tau_R > t$ the price P_t^U of a risk-free CDS can be expressed as a function of π_t. By iterated conditional expectations,

$$P_t^U := \mathbb{E}\left(\Pi(t,T)\big|\mathcal{F}_t^U\right) = \mathbb{E}\left(\mathbb{E}\left(\Pi(t,T)\big|\mathcal{F}_t^O\right)\big|\mathcal{F}_t^U\right) = \sum_{k\in S^X} p(t,k)\pi_t^k. \tag{3.5}$$

Finally, the dynamics of π_t can be described by a K-dimensional SDE system; see Proposition 4.1 below. From this SDE system we may in particular derive an explicit formula for the contagion effects under incomplete information.

In the practical application of the model the process Z is considered as abstract source of information and the current value of π is calibrated from observed prices of traded credit derivatives; see Section 6.1 below.

Note that in both model variants the unconditional joint survival function of τ_R, τ_B and τ_S is given by

$$\mathbb{Q}(\tau_R > t_R, \tau_B > t_B, \tau_S > t_S) = \mathbb{E}\Big(\prod_{i\in\{B,R,S\}} \exp\Big(-\int_0^{t_i} \lambda_i(X_s)ds\Big)\Big)\,,$$

so that the distributions of (τ_B, τ_R, τ_S) coincides in both versions of the model. Therefore any differences in the BCCVA or in the performance of collateralization strategies can be attributed to the different dynamics of CDS spreads. For illustrative purposes we plot typical trajectories of CDS spreads in Figure 5

4. Computation of the BCCVA

4.1. *The case of the full-information model*

In order to evaluate the BCCVA-formula (2.4) we need to determine the joint distribution of τ, ξ and X_τ. This is done in Theorem 4.1 below. The proof of this result relies on the observation that the distribution of the triple (τ, ξ, X_τ) can be expressed as first entry time of the Markov chains (X, H^R) and (X, H) into specific sets[c]. In order to give precise results, we need to specify the generators of these Markov chains.

We assume that the states are ordered in the *inverse lexicographic order.* According to this order a vector $(x_1, \dots, x_n)$ is smaller than $(y_1, \dots, y_n)$ if $x_n < y_n$ or if there is some $k < n$ with $x_{l+1} = y_{l+1}$ for $l \in \{k, \dots, n-1\}$ and with $x_k < y_k$. For example, in the case $K = 2$ the states of the process (X, H^R) are ordered in the following way:

$$(1,0) < (2,0) < (1,1) < (2,1).$$

The transition rate $q_{y,z}$ of (X, H^R) from a state $y = (y_1, y_2)$ to the state $z = (z_1, z_2)$ is given by:

$$q_{y,z} = \begin{cases} w_{y_1\, z_1} & \text{if } y_1 \neq z_1 \text{ and } y_2 = z_2 \\ \lambda_R(y_1) & \text{if } y_1 = z_1, y_2 = 0 \text{ and } z_2 = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence the generator of the process (X, H^R) can be represented by the matrix

$$Q := \begin{pmatrix} W - \Lambda_R & \Lambda_R \\ 0 & W \end{pmatrix}.$$

Here $\Lambda_R = \operatorname{diag}(\lambda_R(1), \dots, \lambda_R(K))$ denotes a diagonal matrix with entries on the main diagonal given by the elements of the vector λ_R. The transition rates and the generator of (X, H) can be determined by analogous considerations.

Theorem 4.1. *Let $t < s$ and $k \in S^X$. Then the following statements hold:*

(a) The distribution of τ_i with $i \in \{B, R, S\}$ is given by

$$\mathbb{Q}\left(\tau_i \le s \middle| X_t = k, H_t^i = 0\right) = 1_{\{\tau_i > t\}} \left(1 - e_k^\top e^{Q_i(s-t)} \boldsymbol{1}_K\right).$$

Here $Q_i := W - \Lambda_i$ where $\Lambda_i = \operatorname{diag}(\lambda_i(1), \dots, \lambda_i(K))$, $\boldsymbol{1}_K = (1, \dots, 1)^\top$ is a column vector of dimension K and e_l denotes the lth unit vector in $\mathbb{R}^K$.

(b) The distribution of the first-to-default time τ can be computed as:

$$\mathbb{Q}\left(\tau \le s | X_t = k, H_t = 0\right) = 1_{\{\tau > t\}} \left(1 - e_k^\top e^{Q_{(1)}(s-t)} \boldsymbol{1}_K\right),$$

where we defined $Q_{(1)} := W - \sum_{j \in \{B,R,S\}} \Lambda_j$.

[c]Distributions which can be represented as the first-entry time of a finite state Markov chain are known as *phase-type distributions.* They have been used in credit risk before; see for instance Herbertsson (2011)

(c) *The probability that obliger $i \in \{B, R, S\}$ defaults first and before time s is:*

$$\mathbb{Q}\left(\tau_i \le s, \xi = i | X_t = k, H_t = 0\right) = 1_{\{\tau_i > t\}} e_k^\top Q_{(1)}^{-1} \left(e^{Q_{(1)}(s-t)} - I\right) \Lambda_i \boldsymbol{1}_K.$$

Here $Q_{(1)}^{-1}$ is the inverse of $Q_{(1)}$.

(d) *The probability that obliger $i \in \{B, R, S\}$ defaults first and that at default the Markov chain is in the state l equals*

$$\mathbb{Q}\left(X_\tau = l, \tau_i \le s, \xi = i | X_t = k, H_t = 0\right) = 1_{\{\tau_i > t\}} e_k^\top Q_{(1)}^{-1} \left(e^{Q_{(1)}(s-t)} - I\right) \Lambda_i e_l.$$

Proof. See Appendix A. □

Corollary 4.1 (Risk-free CDS price under full-information). *The price P_t^O of a risk-free CDS on the reference entity at time t given that $X_t = l$ and $\tau_R > t$ is equal to $1_{\{\tau_R > t\}} p^O(t, l)$, where the function $p^O : [0, T] \times S^X \to \mathbb{R}$ is given by*

$$p^O(t, l) = \left(-\operatorname{LGD}_R e_l^\top Q_R - S e_l^\top\right) (Q_R - rI)^{-1} \left(e^{(Q_R - rI)(T-t)} - I\right) \boldsymbol{1}_K.$$

Moreover, the price of a CDS at $t = 0$ is

$$P_0^O = \left(-\operatorname{LGD}_R \pi_0^\top Q_R - S\pi_0^\top\right) (Q_R - rI)^{-1} \left(e^{(Q_R - rI)(T-t)} - I\right) \boldsymbol{1}_K.$$

Proof. See Appendix A. □

Below we will see that for a suitable function $g\colon [0, T] \times S^X \to \mathbb{R}$, collateralization strategies of the form $C_t = g(t, X_t)$ are optimal in the full-information model. For a generic strategy of this form Theorem 4.1(d) gives the following nearly explicit formula for the BCCVA.

Corollary 4.2 (BCCVA formula under full information). *Assume that for $t < \tau$ the collateralization strategy is of the form $C_t = g(t, X_t)$. Then, given $X_t = j$, the* CCVA *and* CDVA *are given by:*

$$\begin{aligned}
\operatorname{CCVA}_t = \sum_{k \in \{1, \dots, K\}} \int_t^T D(t, s) \Big(&\operatorname{LGD}_S \left(p^O(s, k)^+ - g(s, k)^+\right)^+ \\
&+ \operatorname{LGD}'_S \left(g(s, k)^- - p^O(s, k)^-\right)^+\Big) f_{j,k}^S(s) ds \\
\operatorname{CDVA}_t = \sum_{k \in \{1, \dots, K\}} \int_t^T D(t, s) \Big(&\operatorname{LGD}_B \left(g(s, k)^- - p(s, k)^-\right)^- \\
&+ \operatorname{LGD}'_B \left(p(s, k)^+ - g(s, k)^+\right)^-\Big) f_{j,k}^B(s) ds, .
\end{aligned}$$

Here the functions $f_{j,k}^i$, $i \in \{B, S\}$, are given by

$$f_{j,k}^i(s) := \frac{d}{ds} \mathbb{Q}(\tau \le s, \xi = i, X_\tau = k \mid X_t = k, H_t = 0) = 1_{\{\tau_i > t\}} e_j^\top e^{Q_{(1)}(s-t)} \Lambda_i e_k.$$

Since default correlations enter in the calibration of the model we end this section by a closed-form solution for default correlations evaluated at time $t = 0$. Formally $\varrho_{i,j}^T$, the default correlation of firm i and j, is given by

$$\varrho_{i,j}^T := \operatorname{corr}(1_{\{\tau_i \le T\}}, 1_{\{\tau_j \le T\}}) = \frac{-p_i^T p_j^T}{\sqrt{p_i^T(1-p_i^T)p_j^T(1-p_j^T)}},$$

where $p_i^T = \mathbb{Q}(\tau_i \le T)$, $p_j^T := \mathbb{Q}(\tau_j \le T)$ and $p_{i,j}^T := \mathbb{Q}(\tau_i \le T, \tau_j \le T)$. Applying Theorem 4.1(a) and (b) gives

Corollary 4.3 (Default correlation). *For $i \neq j$ $\varrho_{i,j}^T$ is given by*

$$\varrho_{i,j}^T = \frac{\pi_0(e^{(W-\Lambda_i-\Lambda_j)T})\boldsymbol{1}_K - 1 + p_i^T + p_j^T - p_i^T p_j^T}{\sqrt{p_i^T(1-p_i^T)p_j^T\,.(1-p_j^T)}}.$$

4.2. *The* BCCVA *in the incomplete-information model*

In this section we discuss the computation of the BCCVA under incomplete information. We begin with a formula for the risk-free CDS price. By combining (3.5) and Corollary 4.1 we obtain

Corollary 4.4 (risk-free CDS prices under incomplete information). *Given that $\{\tau_R > t\}$ the value P_t^U of a risk-free CDS at time t equals*

$$P_t^U = p^U(t, \pi_t) := \left(-\mathrm{LGD}_R\, \pi_t^\top Q_R - S\pi_t^\top\right)(Q_R - rI)^{-1}\left(e^{(Q_R-rI)(T-t)} - I\right)\boldsymbol{1}_K.$$

Note that for $t = 0$ one has $P_0^O = P_0^U$; this equality reflects of course the fact that the unconditional distributions of the default times coincide in the two model variants.

Under incomplete information the BCCVA is essentially the value of a portfolio of options on the price P^U of the risk-free CDS. Since P_t^U is a function of π_t, in order to compute the BCCVA one thus needs the form of the dynamics of the process π, and we now recall the relevant results from Frey and Schmidt (2012). We begin with some notation. We denote the optional projection of a process $G = (G_t)_{t\in[0,T]}$ with respect to $\mathbb{F}^U$ by $\widehat{G}$ so that $\widehat{G}_t = \mathbb{E}\left(G_t \middle| \mathcal{F}_t^U\right)$. In particular,

$$\widehat{\lambda}_{t,i} = \mathbb{E}\left(\lambda_i(X_t)\middle|\mathcal{F}_t^U\right) = \sum_{j=1}^K \lambda_i(j)\pi_t^j.$$

$$\widehat{a}_t = \mathbb{E}\left(a(X_t)\middle|\mathcal{F}_t^U\right) = \sum_{j=1}^K a(j)\pi_t^j$$

Using the Levy-characterization of Brownian motion it is easily seen that

$$\mu_t = (\mu_t^1, \ldots, \mu_t^d) \text{ with } \mu_t^i = Z_t^i - \int_0^t \widehat{a(X_s)}ds$$

is a $\mathbb{F}^U$-Brownian motion. Moreover, it is well-known that $\widehat{\lambda}_{t,i}$ is the $\mathbb{F}^U$ default intensity of firm i.

Proposition 4.1 (Kushner-Stratonovich-equation). *The process π is the unique solution of the K-dimensional SDE system*

$$d\pi_t^k = \sum_{i=1}^K w_{ik}\pi_t^i dt + \sum_{j\in\{R,B,S\}} \big(\gamma_j^k(\pi_{t-})\big)^\top d\big(H_t^j + (1-H_t^j)\widehat{\lambda}_{t,j}dt\big) + \big(\alpha^k(\pi_t)\big)^\top d\mu_t\,,$$

$k = 1, \dots, K$, *where*

$$\gamma_j^k(\pi_t) = \pi_t^k \left(\frac{\lambda_j(k)}{\sum_{i=1}^K \lambda_j(i)\pi_t^i} - 1\right) \text{ for } 1 \le j \le K \text{ and}$$
$$\alpha^k(\pi_t) = \pi_t^k \left(a(k) - \sum_{i=1}^K \pi_t^i a(i)\right).$$

The proposition shows that the process π exhibits jump-diffusion dynamics. In particular, π jumps at default times and the jump height of π_t^k at the default of firm j is equal to $\gamma_j^k(\pi_{\tau_j-})$.

Using the proposition we can moreover compute the size of the information-induced contagion effects: the jump in the $\mathbb{F}^U$-default intensity of firm i at the default of firm j equals

$$\widehat{\lambda}_{\tau_j,i} - \widehat{\lambda}_{\tau_j-,i} = \sum_{k=1}^K \lambda_i(k)\pi_{\tau_j-}^k \left(\frac{\lambda_j(k)}{\sum_{l=1}^K \lambda_j(l)\pi_{\tau_j-}^l} - 1\right) = \frac{\mathrm{cov}^{\pi_{\tau_j-}}(\lambda_i,\lambda_j)}{E^{\pi_{\tau_j-}}(\lambda_j)},\ j \neq i. \tag{4.1}$$

An inspection of this formula shows the following.

- Contagion effects are inversely proportional to the instantaneous default risk of the defaulting entity (firm j): a default of an entity with a better credit quality comes as a bigger surprise and the market impact is larger.
- Contagion effects are proportional to the covariance of the default intensities $\lambda_i(\cdot)$ and $\lambda_j(\cdot)$ under the 'a-priori distribution' π_{τ_j-}. In particular, contagion effects are relatively high if the firms have similar characteristics in the sense that the functions $\lambda_i(\cdot)$ and $\lambda_j(\cdot)$ are (almost) linearly dependent.

Proposition 4.1 indicates a method to simulate a trajectory of π. The following high-level algorithm is suggested in Frey and Schmidt (2012).

(1) Generate a trajectory of the Markov chain X.
(2) Generate for the trajectory of X constructed in (i) a trajectory of the default indicator H and the noisy information Z.
(3) Solve the system of SDEs numerically, for instance via a Euler-Maruyama type method.

We close this section with a theoretical result on the relation of the un-collateralized CCVA in the two versions of the model.

Proposition 4.2. *Assume that the CDS contract is un-collateralized, i.e.* $C_t \equiv 0$. *Then the following relationships hold:*

$$\mathrm{CCVA}_0^O \geq \mathrm{CCVA}_0^U \ \ \textit{and} \ \ \mathrm{CDVA}_0^O \geq \mathrm{CDVA}_0^U \ .$$

Proof. We begin with the CCVA. We get, using the definition of the CCVA for the case without collateralization, Jensen's inequality and the relation $P_\tau^U = \mathbb{E}(P_\tau^O \mid \mathcal{F}_\tau^U)$ that

$$\begin{aligned}
\mathrm{CCVA}_0^O &= \mathrm{LGD}_S \, \mathbb{E}\left(1_{\{t<\tau<T\}} 1_{\{\xi=S\}} \left(P_\tau^O\right)^+\right) \\
&= \mathrm{LGD}_S \, \mathbb{E}\left(1_{\{t<\tau<T\}} 1_{\{\xi=S\}} \mathbb{E}\left(\left(P_\tau^O\right)^+ \middle| \mathcal{F}_\tau^U\right)\right) \\
&\geq \mathrm{LGD}_S \, \mathbb{E}\left(1_{\{t<\tau<T\}} 1_{\{\xi=S\}} \left(\mathbb{E}\left(P_\tau^O \middle| \mathcal{F}_\tau^U\right)\right)^+\right) \\
&= \mathrm{LGD}_S \, \mathbb{E}\left(1_{\{t<\tau<T\}} 1_{\{\xi=S\}} (P_\tau^U)^+\right),
\end{aligned}$$

and the last line is obviously equal to CCVA_0^U. A similar reasoning applies to the CDVA. □

The overall relation of the BCCVA in the two model variants is in general unclear, since the BCCVA is the difference of the CCVA and CDVA. In the special case where B is of a much higher credit quality than S the CDVA is almost zero and we have the relation $\mathrm{BCCVA}^O \geq \mathrm{BCCVA}^U$. Similarly, if S is of a much higher credit quality than B one has $\mathrm{BCCVA}^O \leq \mathrm{BCCVA}^U$ (always for the un-collateralized case).

5. Collateralization strategies

Standard collateralization strategies. We consider among others the following collateralization strategies. *No collateralization* corresponds to the strategy $C_t \equiv 0$. The *threshold-collaterization* with initial margin γ and thresholds M_1, $M_2 \geq 0$, labeled C^{γ, M_1, M_2} is given by

$$C_t^{\gamma, M_1, M_2} := \gamma + (P_t - M_1) 1_{\{P_{t-} > M_1\}} + (P_t + M_2) 1_{\{P_t < -M_2\}} \ \forall t \in [0, T \wedge \tau).$$

This strategy is used if B and S want to protect themselves against severe losses, while accepting the possibility of small losses in order to simplify the collateralization process. At the beginning of the contract an initial payment of collateral of the size of γ takes place that can be used as a crude device to account for contagion effects. Additional collateral is only posted if the exposure of one entity exceeds some threshold (M_1 in case of B and M_2 in case of S). Threshold collateralization is quite popular in practice, see Gregory (2010). However, the choice of γ in practice is often based on rules of thumb (compare ICMAs European Repo Council (2012) for Repos), possibly reducing the effectiveness of this strategy. For $\gamma = M_1 = M_2 = 0$ we obtain the special case of *market-value collateralization* C^{market} with $C_t^{\text{market}} = P_t$.

Refined collateralization strategies. In the following we propose refined collateralization strategies that attempt to reduce the overall counterparty-risk exposure of the contracting parties. We use the CCVA to measure the exposure to counterparty risk of B and the CDVA to measure the exposure of S. B and S have obviously conflicting interests: B prefers a collateralization strategy where S posts a large amount of collateral and B posts no collateral and vice versa for B. In order to balance these conflicting interests we propose to choose the 'optimal' collateralization strategy as a minimizer of the function

$$\begin{aligned} m(C) &:= \mathrm{CCVA}_0 + \mathrm{CDVA}_0 && (5.1)\\ &= \mathbb{E}\left(1_{\{\tau<T\}}1_{\{\xi=S\}}D(t,\tau)(\mathrm{LGD}_S(P_\tau^+ - C_{\tau-}^+)^+ + \mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)^+)\right) && (5.2)\\ &+ \mathbb{E}\left(1_{\{\tau<T\}}1_{\{\xi=B\}}D(t,\tau)(\mathrm{LGD}_B(C_{\tau-}^- - P_\tau^-)^- + \mathrm{LGD}'_B(P_\tau^+ - C_{\tau-}^+)^-)\right) && (5.3)\end{aligned}$$

over all $\mathbb{F}$ adapted collateralization strategies C. In the full-information case we let $\mathbb{F} = \mathbb{F}^O$ and thus $P_\tau = P_\tau^O$; in the incomplete-information case we let $\mathbb{F} = \mathbb{F}^U$ and $P_\tau = P_\tau^U$.

The analysis of the full-information model is straightforward. In that case the market value $(P_t^O)_{t\geq 0}$ is continuous at τ_B respectively at τ_S. Therefore counterparty risk can be eliminated completely by choosing the market-value strategy $C_t^{\mathrm{market}} = P_t^O = p^O(t, X_t)$, $t < \tau$, that is $m(C^{\mathrm{market}}) = 0$. The situation is more involved in the incomplete-information model. In that case the jump of π at τ leads to a jump in the market value P_t^U of the CDS at $t = \tau$ and the collateral position cannot be adjusted at that point. Hence one has for the market value strategy $C_t^{\mathrm{market}} = P_t^U = p^U(t, \pi_t)$, $t < \tau$ that $m(C^{\mathrm{market}}) > 0$.

We therefore try to find an alternative optimal strategy under incomplete information. As a first step we simplify the function m by conditioning on $\mathbb{F}_{\tau-}$. It is well-known that τ is $\mathcal{F}_{\tau-}$ measurable and that for any predictable process L the random variable L_τ is $\mathcal{F}_{\tau-}$ measurable; see Protter (2005), Sec III.2. Moreover, one has for $j \in \{R, B, S\}$ that

$$\mathbb{Q}\left(\xi = j | \mathcal{F}_{\tau-}\right) = \frac{(\widehat{\lambda_j})_{\tau-}}{\sum_{i\in\{B,R,S\}}(\widehat{\lambda}_i)_{\tau-}} =: d_j(\pi_{\tau-}) \tag{5.4}$$

We begin with the CCVA component of m. By conditioning on $\mathcal{F}_{\tau-}$ we get that (5.2) equals

$$\mathbb{E}\left(1_{\{\tau\leq T\}}D(t,\tau)\mathbb{E}\left(1_{\{\xi=S\}}\left(\mathrm{LGD}_S(P_\tau^+ - C_{\tau-}^+)^+ + \mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)^+\right)\big|\mathcal{F}_{\tau-}\right)\right) \tag{5.5}$$

In the sequel we use the notation

$$x_S := x_S(\tau, \pi_{\tau-}) = p^U\left(\tau, \pi_{\tau-} + \mathrm{diag}\left(\gamma_S^1, \ldots, \gamma_S^K\right)\pi_{\tau-}\right) \tag{5.6}$$

to denote the price of the CDS immediately after the default of S; similarly, $x_B := x_B(\tau, \pi_{\tau-})$ denotes the price of the CDS immediately after the default of B. Now note that $1_{\{\xi=S\}}P_\tau^+ = x_S^+$. Hence, using (5.4), the inner conditional expectation in

(5.5) is given by $\left(\mathrm{LGD}_S(x_S^+ - C_{\tau-}^+)^+ + \mathrm{LGD}_S^{'}(C_{\tau-}^- - x_S^-)^+\right)d_S$, and (5.5) equals

$$\mathbb{E}\left(1_{\{\tau\le T\}}D(t,\tau)\left(\mathrm{LGD}_S(x_S^+ - C_{\tau-}^+)^+ + \mathrm{LGD}_S^{'}(C_{\tau-}^- - x_S^-)^+\right)d_S\right).$$

Similarly we get that (5.3), the CDVA component of m, is equal to

$$\mathbb{E}\left(D(t,\tau)(\mathrm{LGD}_B(C_{\tau-}^- - x_B^-)^- + \mathrm{LGD}_B'(x_B^+ - C_{\tau-}^+)^-)d_B\right).$$

Define now the 'infinitesimal loss function'

$$\begin{aligned} l(t,\pi,c) = &\left(\mathrm{LGD}_S(x_S(t,\pi)^+ - c^+)^+\right) + \mathrm{LGD}_S'(c^- - x_S(t,\pi)^-)^+\right) d_S(\pi) \\ &+ \left(\mathrm{LGD}_B(c^- - x_B(t,\pi)^-)^- + \mathrm{LGD}_B'(x_B(t,\pi)^+ - c^+)^-\right) d_B(\pi). \end{aligned}$$

The above computations show that m can be written in the form $m(C) = \mathbb{E}\left(D(t,\tau)l(\tau,\pi_{\tau-},C_{\tau-})\right)$. Suppose now that we find an $\mathbb{F}^U$-adapted $RCLL$-process C^* such that a.s.

$$C_t^*(\omega) \in \operatorname{argmin}\{l(t,\pi_t(\omega),c)\colon c\in\mathbb{R}\}.$$

Then C^* is an optimal collateralization strategy - a minimizer of $m(\cdot)$ - in the incomplete-information model. This leads to the following proposition.

Proposition 5.1. *Denote by $x_S = x_S(\pi_t)$ and $x_B = x_B(\pi_t)$ the risk-free CDS price at time t given $\tau = t, \xi = S$ respectively $\tau = t, \xi = B$ (see (5.6)) and let $d_S = d_S(\pi_t) = \frac{\widehat{(\lambda_S)}_t}{\sum_{i\in\{B,R,S\}}\widehat{(\lambda_i)}_t}$ and similarly for d_B. Consider an $\mathbb{F}^U$-adapted RCLL process C^*. C^* is a minimizer of the function m under incomplete information if and only if the following relations hold $\mathbb{Q}$-a.s. for $t<\tau$:*

$$C_t^* = \begin{cases} x_S & \text{if } 0\le x^B \le x^S, LGD_B'd_B < LGD_Sd_S \\ x_B & \text{if } 0\le x^B \le x^S, LGD_B'd_B > LGD_Sd_S \\ x_S & \text{if } x_B \le x_S \le 0, LGD_Bd_B < LGD_S'd_S \\ x_B & \text{if } x_B \le x_S \le 0, LGD_B'd_B > LGD_Sd_S \\ \operatorname{argmin}\{l(\tau,\pi_t,c)\colon c = x_B, 0, x_S\} & \text{if } x_B < 0 < x_S \end{cases}$$

$$C_t^* \in \begin{cases} [x_B, x_S] & \text{if } 0\le x_B \le x_S, LGD_B'd_B = LGD_Sd_S \\ [x_B, x_S] & \text{if } x_B \le x_S \le 0, LGD_Bd_B = LGD_S'd_S \\ [x_S, x_B] & \text{if } x_S \le x_B. \end{cases}$$

In particular we have $l(t,\pi,C_t) = 0$ for $x^S \le x^B$. Moreover, it holds that always $C_t^* \in [\min\{x^S,x^B\},\max\{x^S,x^B\}]$.

Proof. The proof mainly relies on the preceding arguments. In order to find an optimal strategy we have to minimize the function $c\mapsto l(\tau,\pi,c)$. This is a piecewise linear function, which converges to ∞ for $c\to\pm\infty$ and fixed t,π. Therefore a minimum exists that can be found by a case-by-case analysis. Here we will only discuss the case $0 < x_B < x_S$. The other cases can be treated in a similar manner. If $0 < x_B < x_S$, l becomes:

$$l(t, \pi, c) = (\mathrm{LGD}_S (x_S - c^+)^+ + \mathrm{LGD}'_S\, c^-) d_S + (\mathrm{LGD}'_B (x_B - c^+)^-) d_B$$

In this case l is decreasing in the interval $(-\infty, x_B]$ and increasing in $[x_S, \infty)$. Therefore the optimal c lies in $[x_B, x_S]$. For $c \in [x_B, x_S]$, l is given by:

$$l(\tau, \pi_{\tau-}, c) = c(LGD'_B d_B - LGD_S d_S) + LGD_S x_S d_S - LGD'_B x_B d_B.$$

Therefore the result follows. The other claims can be established by a similar case-by-case analysis. □

Comments. We will see below that this strategy performs well even under incomplete information. However, if $\mathbb{Q}(x^B(\pi_t) > x^S(\pi_t)) > 0$ there remains some risk, that is $m(C^*) > 0$. This remaining risk is due to the fact that in an inhomogeneous portfolio the size of the contagion effects at τ depends on the identity of the defaulting firm which cannot be predicted given the information contained in $\mathcal{F}_{\tau-}$.

Note that the refined strategy depends on d_B, d_S, and, most importantly, on the market value x_S and x_B of the risk-free CDS after the default of S or B and hence on the size of contagion effects. While these quantities can be easily computed within a given calibrated reduced-form credit risk model with contagion such as the model of Frey and Schmidt (2012) used here, they do depend on the specific form of the model. In order to obtain a 'model-independent' version of C^* one could start from some ad-hoc assumption on the size of contagion effects, based perhaps on historical data analysis and on the qualitative insights that derive from Equation (4.1); d_S and d_B could be estimated by $d_S = S_S/(S_B + S_S + S_R)$, $d_B = S_B/(S_B + S_S + S_R)$ where S_B, S_R and S_S the fair CDS spread for B, R and S observed on the market.

6. Numerical Experiments

In this section we discuss the results of a number of numerical experiments.

6.1. *Setup and Calibration*

We consider a Markov chain X with $K = 8$ states. We assume that X exhibits next-neighbor dynamics, so that only the values on the main diagonal and on the first off-diagonal of the generator matrix W may be different from zero. For our simulations we set the entries on the off-diagonal equal to 0.25, which means that the Markov chain jumps on average 2 times per year. We put the short-rate equal to $r = 0.015$. Throughout the study we assume that $\mathrm{Rec}_B = \mathrm{Rec}_S = \mathrm{Rec}_R = 0.5$ and $\mathrm{Rec}'_B = \mathrm{Rec}'_S = 0.75$. Qualitative results of the simulation study did not change, when other parameters were used.

We calibrate the model to given risk-free CDS spreads and default correlations for R, B and S. We consider five different scenarios, labeled *Base; Base 2; Symmetric; Risky protection buyer; Risky protection seller.* These scenarios differ mainly with respect to the relative riskiness of the firms involved in the CDS contract; their choice serves to illustrate the impact of the relative riskiness of the different firms on credit value adjustments. The fair CDS spreads (in basis points) and default correlations (in percent) corresponding to these scenarios can be found in Table 6.1.

Table 1. Risk scenarios: CDS-spreads are given in base points, default correlations in percentage points

Name of scenario	B	R	S	ρ_{BR}	ρ_{BS}	ρ_{RS}
Base	50	1000	500	2.0	1.5	5.0
Base2	500	1000	50	5.0	1.5	2.0
Symmetric	500	1000	500	5.0	3.0	5.0
Risky PB	1000	500	50	5.0	2.0	1.5
Risky PS	50	500	1000	1.5	2.0	5.0

In this context model model calibration amounts to finding the initial distribution of the Markov chain π_0 and the parameters $\lambda_B, \lambda_R, \lambda_S$. For calibration purposes we use a modification of the algorithm presented in Frey and Schmidt (2012); since the focus of this paper is not on model calibration we omit the details. All in all the calibration procedure performed well, with very small errors for CDS spreads (absolute errors are less than 0.5 bp) and acceptable results for default correlations (relative errors are around 3%). The calibrated values of π_0 and of λ_B, λ_R and λ_S can be found in the appendix, Table 2. Note in particular that the calibrated functions $\lambda_B(\cdot)$, $\lambda_S(\cdot)$ and $\lambda_R(\cdot)$ are increasing in x. In the incomplete-information model we moreover need to choose the parameters $a(1), \dots, a(K)$. We took $a = c{*}b$, where $b = [-1.75, -1.25, -0.75, -0.25, 0.25, 0.75, 1.25, 1.75]$ and where $c \geq 0$ If not mentioned otherwise, c is taken equal to one.

6.2. *Results*

No collateralization

The main findings regarding the qualitative behavior of the CCVA and CDVA in the case of no collateralization can be summarized as follows.

a) The size of the credit value adjustments depends largely on the relative riskiness of the firms. In particular, the un-collateralized CCVA is comparatively high if the *first-to-default probability* $\mathbb{Q}(\tau \leq T, \xi = S)$ is relatively large; similarly, the un-collateralized CDVA is comparatively high if $\mathbb{Q}(\tau \leq T, \xi = B)$ is relatively large. This can be seen by comparing the size of the value adjustments for the Base and Base2 scenarios or the RiskyPB and the RiskyPS scenarios in Tables 5 and 6: as shown in Table 4, $\mathbb{Q}(\tau \leq T, \xi = S)$ is relatively large in the Base and the RiskyPS scenarios, leading to a high CCVA; similarly, $\mathbb{Q}(\tau \leq T, \xi = B)$ is relatively large in the Base2 and the RiskyPB scenarios, leading to a high CDVA. Note that the first-to-default probabilities are identical in both versions of the model. They are largely driven by the (relative) riskiness of the three firms as given by the risk-free CDS spread in the three scenarios.

b) Without collateralization we have $\mathrm{CCVA}^U < \mathrm{CCVA}^O$ and $\mathrm{CDVA}^U < \mathrm{CCVA}^O$, as predicted by Proposition 4.2. The differences between the model variants decreases with decreasing observation noise, that is for higher values of the parameter c in the definition of the function a, as can be seen by inspection of Table 7.

c) In both versions of the model there is clear evidence for so-called *wrong-way risk*: the conditional default probability of the reference entity given an early default of the protection seller is much higher than the unconditional default probability of R. In the full-information case this can be seen from Table 3 which gives the distribution of X_τ for the case $\xi = B$ and $\xi = S$. Clearly,the Markov chain tends to be in a higher state (compare the high probabilities for x_8) at a default. Hence prices of CDS at τ tend to be high too. Figure 1 shows that the behavior of X_τ is passed onto π_τ in the sense that π_τ^j has rather small realizations if j is small and high realizations if j is high. This is confirmed by the shapes of the empirical distributions of π^1 and π^8 (recall that π^1 respectively π^8 gives the conditional probability of being in a good state (low default intensities) respectively in a bad state (high default intensities).) Findings b) and c) show that while event correlations and wrong way risk are more crucial for determining un-collateralized credit value adjustments, contagion effects per se are not so important (but they will matter for the performance of collateralization strategies). The possibility to distinguish between event correlation and default contagion is thus clearly an advantage of our setting.

Collateralization

We go on with the analysis of various collateralization strategies. Since collateralization is only relevant on paths where $\tau < T$ and where $\xi \in \{B, S\}$, we illustrate the performance of collateralization strategies by plotting the conditional empirical distribution function of the following random variables:

$$L_B(C) := 1_{\{\xi=S\}}\big(\mathrm{LGD}_S(P_\tau^+ - C_{\tau-}^+)^+ + \mathrm{LGD}'_S(C_{\tau-}^- - P_\tau^-)^+\big).$$
$$L_S(C) := D(t,\tau)1_{\{\xi=B\}}\big(\mathrm{LGD}_B(C_{\tau-}^- - P_\tau^-)^- + \mathrm{LGD}'_B(P_\tau^+)^- - C_{\tau-}^+\big).$$

given that $\{\tau \le T, \xi \in \{B, S\}\}$. Note that $L_B(C)$ gives discounted loss to B that arises from an early default of S, whereas $L_S(C)$ gives the discounted loss to S that arises from an early default of B.

We analyze strategies of the following type:

- Threshold-collateralization with initial margin γ and thresholds $M_1 = M_2 := M$, denoted $C^{\gamma,M}$;
- Market collateralization $C^{\mathrm{market}} = C^{0,0}$ in the full-information model;
- The strategy C^* derived in Proposition 5.1 for the incomplete-information model.

Our findings can be summarized as follows:

a) Threshold collateralization with $\gamma = 0$ is very effective in the complete-information model. For a threshold $M > 0$ counterparty risk is largely reduced as can be seen from Table 8. Counterparty credit risk even vanishes completely for $M = 0$. Moreover, losses are bounded when threshold-collateralization is used. This can be seen from Figure 2 which displays the empirical cdf of L_B given $\tau \leq T$ and $\xi \in \{B, S\}$ in the complete information model for different scenarios.

b) Under incomplete information the performance of threshold collateralization with $\gamma = 0$ and threshold M is not fully satisfactory. The main reason is the fact that because of the contagion effects threshold collateralization systematically underestimates the market value of the CDS at τ which leads to losses for the protection buyer in case that $\xi = S$. As a consequence we observe high values for the CCVA in scenarios such as the Base scenario where $\mathbb{Q}(\tau \leq T, \xi = S)$ is comparatively high, compare Table 9. The losses of the protection seller on the other hand are always smaller than the threshold M. This behavior can be seen from Figures 3 and 4 where the conditional cdf of L_B and L_S is plotted in various scenarios.

A nonzero initial margin γ can improve the performance of threshold collateralization in scenarios where the credit quality of B is much better than the credit quality of S as in the Base scenario. In that case $\mathbb{Q}(\xi = S \mid \tau \leq T, \xi \in \{B, S\})$ is close to one and one essentially knows that $\xi = S$ in case of an early default. Consequently it is possible to hedge a large part of the contagion effects by choosing a positive initial margin γ. This can be seen from Figure 6 where $m(C^{\gamma,M})$ is plotted in the Base scenario for various values of γ and M. In a symmetrical scenario where B and S have similar credit quality on the other hand, the identity of the first defaulting firm cannot be predicted and choosing a nonzero initial margin does not help much to improve the effectiveness of threshold collateralization, as can be seen from Figure 7. This is clear intuitively: a large initial margin $\gamma > 0$ will lead to a loss for S in case that $\xi = B$ because of re-hypothecation; on the other hand for $\gamma \leq 0$ there will be a loss for B in case that $\xi = S$ because of contagion effects, and neither of the two cases can be 'ruled out' a-priori because B and S have similar credit quality.

c) The refined strategy C^* on the other hand performs well under incomplete information and reduces counterparty risk substantially as can be seen from Table 10 where various credit value adjustments and the value of $m(C)$ are given. The strategy is particularly effective in scenarios where the credit quality of B is higher than the credit quality of S so that $x_S \leq x_B$ such as the Base scenario and the Risky-PS scenario. In particular the refined strategy leads to a value of $m(C^*) = 0$ (complete elimination of counterparty risk) in the symmetric scenario case where threshold collateralization did not perform particularly well. On the other hand C^* does not fully eliminate counterparty risk in scenarios where the credit quality of S is worse than the credit quality of B such as the Base2 and the Risky-PB scenario, as is evident from Table 10 and Figures 8 and 9). However, even in these scenarios the probability that some party suffers a large loss is fairly small.

Of course, the superior performance of the refined collateralization strategy is related to the fact that in our framework the quantities x_B and x_S are known exactly. In real markets on the other hand the size of contagion effects cannot be predicted exactly so that the practical implementation of this strategy is subject to a certain degree of *model risk.* Nonetheless our simulations show that refined collateralization strategies that account for contagion effects have the potential to reduce counterparty credit risk significantly in scenarios where there is a non-negligible probability that the protection seller defaults first.

Acknowledgments

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Appendix A. Proofs

Proof. (Proof of Theorem 4.1) By symmetry, it suffices to consider the case $i = B$. a) The default time τ_B is the time, at which the Markov chain (X, H^B) first enters the absorbing set $A = \{(1,1), \dots, (K,1)\}$ and leaves the set $A^c := \{(1,0), \dots, (K,0)\}$. Hence we get:

$$\begin{aligned} \mathbb{Q}\left(\tau_B > s \middle| X_t = k, H_t^B = 0\right) &= \mathbb{Q}\left((X_s, H_s^B) \in A^c \middle| X_t = k, H_t^B = 0\right) \\ &= 1_{\{\tau_B > t\}}(e_k^\top, 0) e^{Q(s-t)} (\mathbf{1}_K^\top, 0)^\top \end{aligned}$$

Here Q denotes the generator of (X, H^B). Q^n is of the form:

$$Q^n = \begin{pmatrix} W - \Lambda_B & \Lambda_B \\ 0 & W \end{pmatrix}^n = \begin{pmatrix} (W - \Lambda_B)^n & * \\ 0 & * \end{pmatrix} = \begin{pmatrix} Q_B^n & * \\ 0 & * \end{pmatrix}.$$

Therefore the entries in the upper left part of the matrix exponential $e^{Q(s-t)}$ are given by $e^{Q_B(t-s)}$ and we can conclude:

$$\mathbb{Q}\left(\tau_B > s \middle| X_t = k, H_t^B = 0\right) = 1_{\{\tau_B > t\}} e_k^\top e^{Q_B(s-t)} \mathbf{1}_K.$$

b) The default times are conditionally independent doubly stochastic random times, and hence the first-to-default time exhibits an intensity, which is given by the sum of the individual intensities (see McNeil, Frey and Embrechts (2005), Lemma 9.36) and the result follows from a).

c) We consider the Markov chain $\Psi_t = (X, H^B, H^R, H^S)_{\tau \wedge t}$ (the chain stopped at the first default time. Ignoring the states where more than one company defaults a(and which can therefore never be reached by Ψ), the infinitesimal generator of Ψ

is given by:

$$\bar{Q} = \begin{pmatrix} W - \sum_{j\in\{B,R,S\}} \Lambda_j & \Lambda_B & \Lambda_R & \Lambda_S \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

The protection buyer B defaults first and before time s if and only if the stopped Markov chain Ψ is in the set $\tilde{A} := \{(1,1,0,0),\ldots,(K,1,0,0)\}$ at time s. Therefore:

$$\begin{aligned}\mathbb{Q}\left(\tau \le s, \xi = B \middle| X_t = k, H_t = (0,0,0)\right) &= \mathbb{Q}\left(\Psi_s \in \tilde{A} \middle| \Psi_t = (k,0,0,0)\right) \\ &= 1_{\{\tau>t\}}(e_k^\top, 0)e^{\bar{Q}(s-t)}(0, \mathbf{1}_K^\top, 0, 0)^\top.\end{aligned}$$

So we have to compute the entries of a submatrix of the matrix exponential $e^{\bar{Q}(s-t)}$. Since the n-th power of the matrix $\bar{Q}(s-t)$ is given by ($n > 0$):

$$(\bar{Q}(s-t))^n = (s-t)^n \begin{pmatrix} Q_{(1)}^n & Q_{(1)}^{n-1}\Lambda_B & Q_{(1)}^{n-1}\Lambda_R & Q_{(1)}^{n-1}\Lambda_S \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

the relevant submatrix is given by

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{Q_B^{n-1}}{n!}(s-t)^n \Lambda_B &= Q_{(1)}^{-1}\left(\sum_{n=0}^{\infty} \frac{Q_{(1)}^{n-1}}{n!}(s-t)^n - I\right)\Lambda_B \\ &= Q_{(1)}^{-1}\left(e^{Q_{(1)}(s-t)} - I\right)\Lambda_B,\end{aligned}$$

and the claim follows.

d) The result follows from similar considerations as in c). □

Proof. (Proof of Corollary 4.1) The conditional distribution of τ given $X_t = k$ and $\tau_i > t$ exhibits the density $u \mapsto -1_{\{\tau_i>t\}} e_k^\top Q_i e^{Q_i(u-t)}\mathbf{1}_K$. Therefore, value of the default leg is given by

$$\begin{aligned}\mathbb{E}\left(\mathrm{LGD}_R\, 1_{\{t<\tau_R\le T\}} D(t,\tau_R) \middle| \mathcal{F}_t^O\right) &= \int_t^T -\mathrm{LGD}_R\, D(t,u) e_k^\top Q_R e^{Q_R(u-t)} 1_K du \\ &= -\mathrm{LGD}_R\, e_k^\top Q_R \int_t^T e^{(Q_R - rI)(u-t)} du 1_K \\ &= -\mathrm{LGD}_R\, e_k^\top Q_R \left[(Q_R - rI)^{-1} e^{(Q_R-rI)(u-t)}\right]_{u=t}^{u=T} 1_K \\ &= -\mathrm{LGD}_R\, e_k^\top Q_R (Q_R - rI)^{-1}\left[e^{(Q_R-rI)(T-t)} - I\right] 1_K.\end{aligned}$$

Furthermore we get for the premium leg:

$$\begin{aligned}\mathbb{E}\left(\int_t^T SD(t,u) 1_{\{\tau_R>u\}} du \middle| \mathcal{F}_t^O\right) &= \int_t^T SD(t,u)\mathbb{E}\left(1_{\{\tau_R>u\}} \middle| \mathcal{F}_t^O\right) du \\ &= \int_t^T S e_k^\top e^{(Q_R-rI)(u-t)} 1_K du \\ &= S e_k^\top (Q_R - rI)^{-1}\left[e^{(Q_R-rI)(u-t)}\right]_{u=t}^{u=T} 1_K \\ &= S e_k^\top (Q_R - rI)^{-1}\left[e^{(Q_R-rI)(T-t)} - I\right] 1_K. \quad \square\end{aligned}$$

Appendix B. Figures and Tables

Table 2. Results of model calibration for the base scenario

state	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
π_0	0.0810	0.0000	0.2831	0.0548	0.0000	0.0000	0.0000	0.5811
λ_B	0.0000	0.0010	0.0027	0.0040	0.0050	0.0059	0.0091	0.0195
λ_R	0.0031	0.0669	0.1187	0.1482	0.1687	0.1855	0.2393	0.3668
λ_S	0.0007	0.0245	0.0482	0.0627	0.0732	0.0818	0.1108	0.1840

Table 3. Distribution of the Markov chain at τ in the base scenario for $\xi = B$ and $\xi = S$.

state	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8
$\xi = B$	0.0001	0.0144	0.0740	0.0500	0.0208	0.0221	0.0982	0.7203
$\xi = S$	0.0011	0.0309	0.1188	0.0713	0.0277	0.0279	0.1074	0.6149

Table 4. The first-to-default probabilities for different scenarios

scenario	B	R	S
Base	0.0293	0.4238	0.2463
Base2	0.2463	0.4238	0.0293
Symmetric	0.1851	0.3972	0.1851
RiskyPB	0.4238	0.2463	0.0293
RiskyPS	0.0293	0.2463	0.4263

Table 5. Valuation adjustments for the complete-information model without collateralization

scenario	CCVA	CDVA	BCCVA
Base	94	1	92
Base2	10	26	-16
Symmetric	74	5	68
RiskyPB	6	45	-39
RiskyPS	115	1	114

Table 6. Valuation adjustments for the incomplete-information model without collateralization

scenario	CCVA	CDVA	BCCVA
Base	83	1	82
Base2	9	15	-6
Symmetric	72	4	68
RiskyPB	6	27	-21
RiskyPS	97	1	96

Table 7. Valuation adjustments under incomplete information for different values of the parameter c (low values of c correspond to a high observation noise) in the base scenario

noise parameter	CCVA	CDVA	BCCVA
$c = 0$	68	0	68
$c = 1$	83	1	82
$c = 2$	89	1	88
$c = 5$	92	1	90

Table 8. Valuation adjustments in the complete-information model with threshold-collateralization with $\gamma = 0$ in the Base scenario

threshold	CCVA	CDVA	BCCVA
$M = 0$	0	0	0
$M = 0.02$	16	0	15
$M = 0.05$	38	1	37
no coll	93	1	92

Table 9. Valuation adjustments in the incomplete-information model with threshold-collateralization with $\gamma = 0$ in Base-scenario

threshold	CCVA	CDVA	BCCVA
$M = 0$	35	0	35
$M = 0.02$	45	0	45
$M = 0.05$	60	0	60
no coll	83	1	82

Table 10. Valuation adjustments in the incomplete-information model with refined collateralization strategy C^*. Note that $m(C^*)$ is small in all scenarios and that $m(C^*) = 0$ in the Base- and RiskyPS scenarios where $x_S \leq x_B$.

scenario	CCVA	CDVA	BCCVA	$m(C^*) = \text{CCVA} + \text{CDVA}$
Base	0	0	0	0
Base2	1	3	-2	4
Symmetric	0	0	0	0
RiskyPB	1	5	-5	6
RiskyPS	0	0	0	0

Fig. 1. Empirical cdf of π_τ in the Base scenario given $\tau \leq T$ and $\xi = S$. Note that π^8_τ (the probability attributed to the worst state) tends to be large.

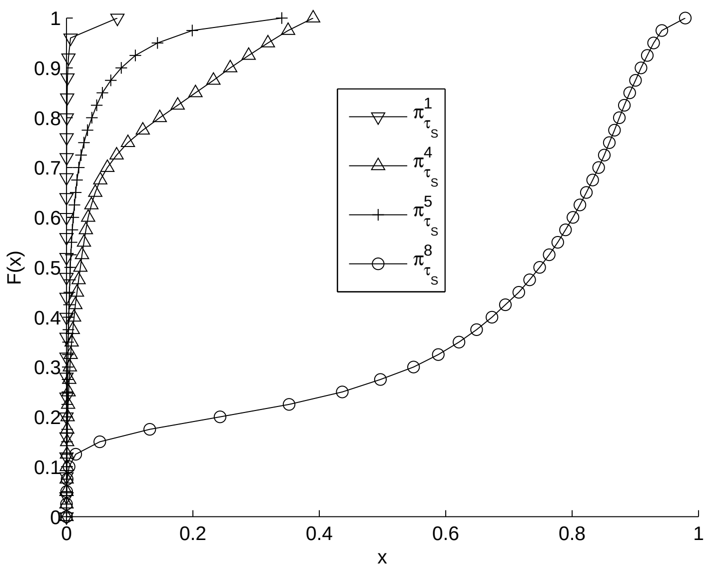


Fig. 2. Empirical cdf of L_B for different threshold-collateralization strategies with $\gamma = 0$ in the Base scenario in the complete-information model given $\tau \leq T$ and $\xi \in \{B, S\}$. Note that without collateralization the probability that L_B is large is quite high since in the base scenario $\mathbb{Q}(\xi = S \mid \tau \leq T, \xi \in \{B, S\}) = 0.245/(0.245 + 0.029) \approx 1$ (see Table 4). We can see that threshold collateralization reduces counterparty credit risk very effectively in that case.

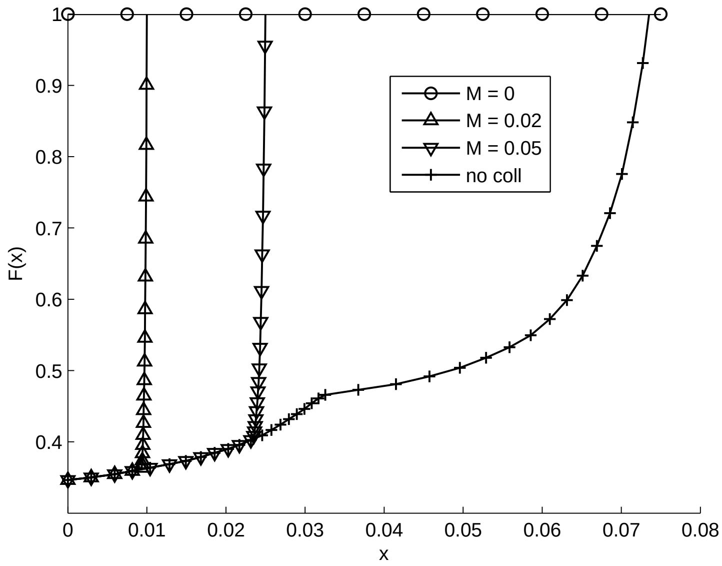


Fig. 3. Empirical cdf of L_B for different threshold-collateralization strategies with $\gamma = 0$ in the Base scenario in the incomplete-information model given $\tau \leq T$ and $\xi \in \{B, S\}$. In that case threshold collateralization with $\gamma = 0$ is not very effective: even for $M = 0$ there is roughly a 20% probability that L_B exceeds 300bp.

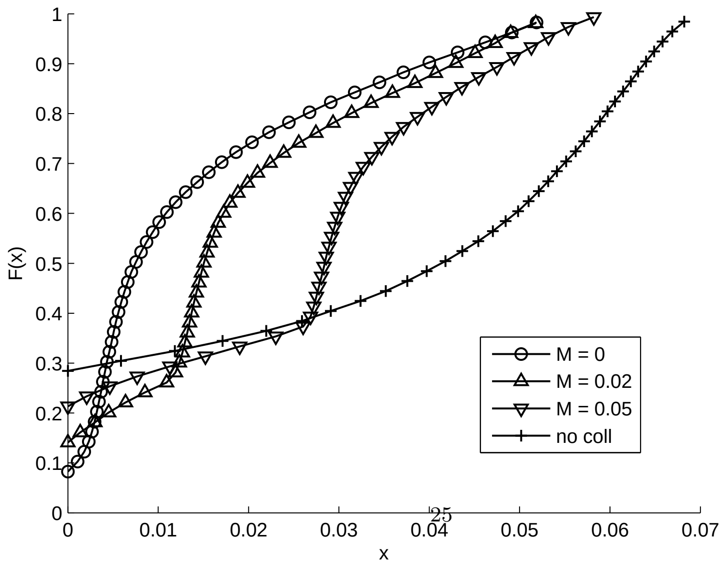


Fig. 4. Empirical cdf of L_S using threshold-collateralization for the Base2 scenario in the incomplete-information model given $\tau \leq T$ and $\xi \in \{B, S\}$. In this scenario $\mathbb{Q}(\xi = B \mid \tau \leq T, \xi \in \{B, S\})$ is close to one so that threshold collateralization is quite effective even under incomplete information.

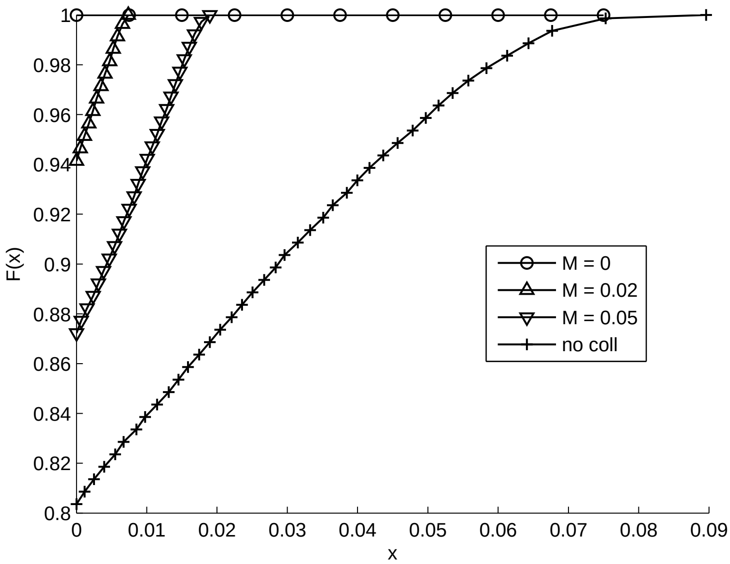


Fig. 5. Trajectories of the fair CDS spread in the complete and incomplete information model.

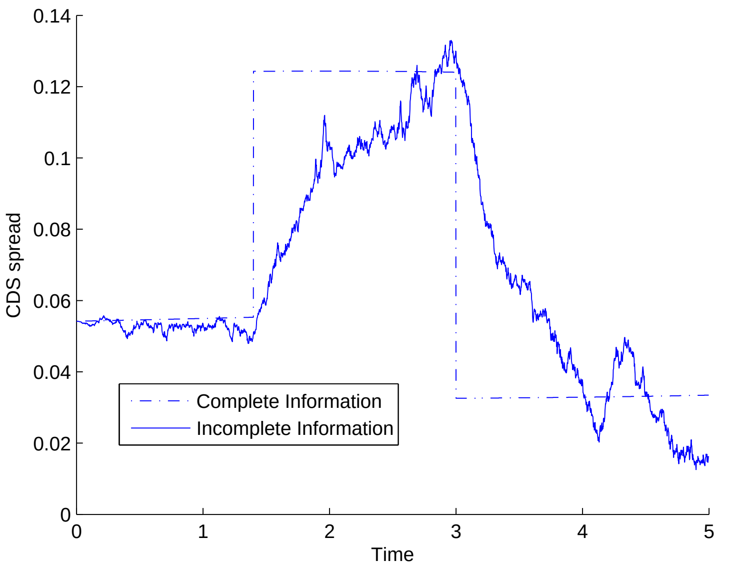


Fig. 6. Graph of $m(C^{\gamma,M})$ (sum of CCVA and CDVA) under incomplete information for the threshold strategy $C^{\gamma,M}$ for varying values of the initial margin γ and the threshold M in the Base scenario. The function $m(C^{\gamma,M})$ is minimal for $M = 0$ and a positive initial threshold $\gamma^* \approx 0.12$ leading to an optimal value $m(C^{\gamma^*,0}) = 3$bp, so that counterparty risk can in effect be mitigated by a proper choice of the initial margin.

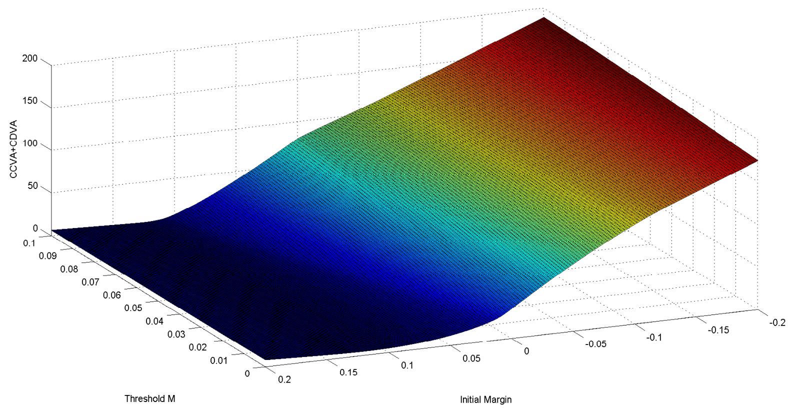


Fig. 7. Graph of $m(C^{\gamma,M})$ (sum of CCVA and CDVA) under incomplete information for the threshold strategy $C^{\gamma,M}$ for varying values of the initial margin γ and the threshold M in the symmetric scenario. The function $m(C^{\gamma,M})$ is minimal for $M = 0$ and a small initial threshold $\gamma^* \approx 0.01$. Note that $m(C^{\gamma^*,0}) = 15$bp whereas for the refined strategy from Proposition 5.1 one has $m(C^*) = 0$.

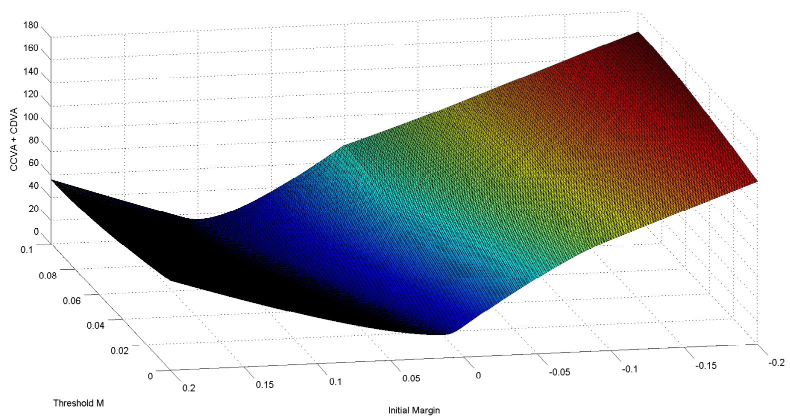


Fig. 8. Empirical cdf of L_B given $\tau \leq T$ and $\xi \in \{B, S\}$ under incomplete information for the refined strategy in various scenarios. Note that in the Base, Symmetric and the RiskyPS scenario $L_B = 0$ a.s., that is counterparty risk for B is eliminated completely by the strategy. In the other scenarios some small degree of counterparty risk remains.

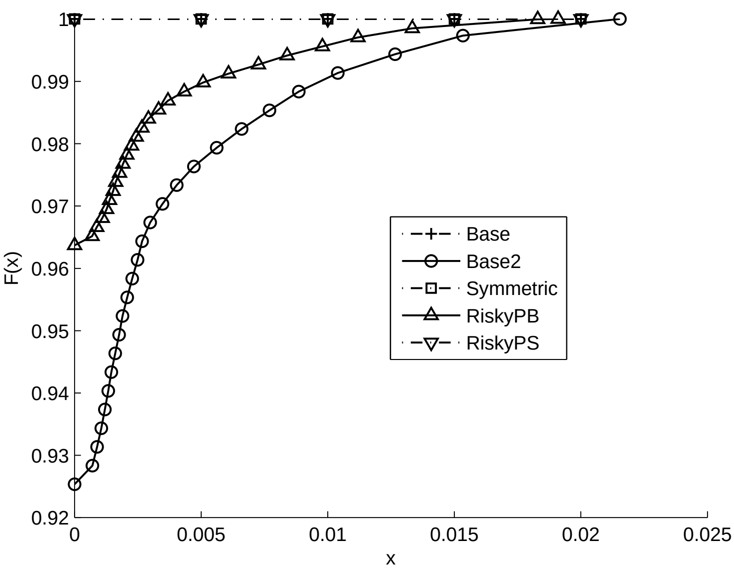
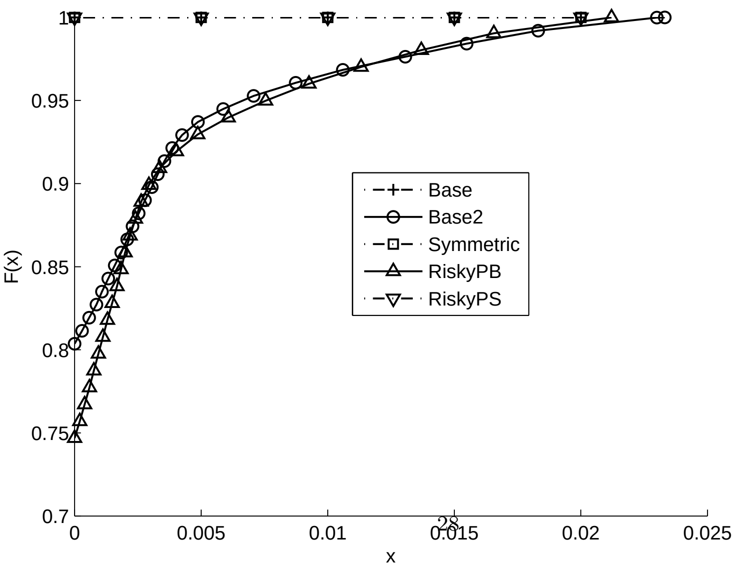


Fig. 9. Empirical cdf of L_S given $\tau \leq T$ and $\xi \in \{B, S\}$ under incomplete information for the refined strategy in various scenarios. Note that in the Base, Symmetric and the RiskyPS scenario $L_S = 0$ a.s., that is counterparty risk for S is eliminated completely by the strategy. In the other scenarios some small degree of counterparty risk remains.



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